NOTES ON THE THEORY OF OSCILLATING CURRENTS.

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§ 1. *Introduction.*

THE object of the following article is to present a short outline sketch of a modification of the method of complex imaginary quantities, applied to oscillating currents. Such oscillating currents have frequently been considered as ordinary alternating currents of very high frequency, and treated as such, while the essential differences between alternating and oscillating currents have been overlooked. An electric current varying periodically between constant maximum and minimum values, that is, in equal time intervals repeating the same values, is called an alternating current if the arithmetic mean value equals zero; and is called a pulsating current if the arithmetic mean value differs from zero. Alternating currents have found very extensive application for light and power. Pulsating currents are the currents given by open coil arc-light machines, or by the superposition of alternating and continuous currents, etc.

Assuming the wave as a sine curve, or replacing it by the equivalent sine wave, the alternating current is characterized by the period or the time of one complete cyclic change, and the amplitude or the maximum value of the current. Period and amplitude are constant in the alternating current.

A very important class are the currents of constant period, but geometrically varying amplitude, that is, currents in which the amplitude of each following wave bears to that of the preceding wave a constant ratio. Such currents consist of a series of waves of constant length, decreasing in amplitude, that is in strength, in constant proportion. They are called oscillating currents in analogy with mechanical oscillations, for instance of the pendulum, in which the amplitude of the vibration decreases in constant proportion.

Since the amplitude of the oscillating current varies, constantly decreasing, the oscillating current differs from the alternating current in so far that it starts at a definite time, and gradually dies out, reaching zero value theoretically at infinite time, practically in a very short time, short even in comparison with the time of one alternating half-wave. Characteristic constants of the oscillating current are the period T or frequency $N=\frac{1}{T}$, the first amplitude and the ratio of any two successive amplitudes, the latter being called the decrement of the wave. The oscillating current will thus be represented by the product of a periodic

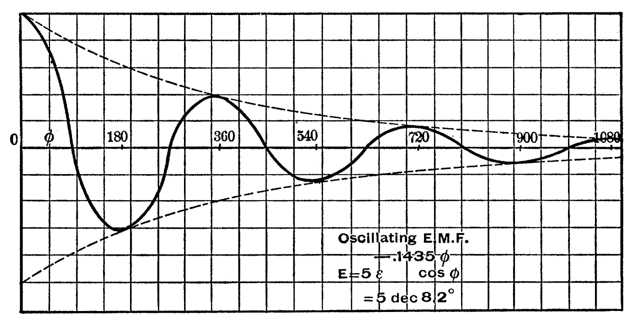

Fig. 1.

function, and a function decreasing in geometric proportion with the time. The latter is the exponential function A^{f-gt}.

Thus, the general expression of the oscillating current is

$$C=A^{f-gt}\cos(2\pi Nt-\hat{\omega}),$$

since $$A^{f-gt}=A^{f}A^{-gt}=c\epsilon^{-bt}.$$

Where ϵ = basis of natural logarithms, the current may be expressed

$$C=c\epsilon^{-bt}\cos(2\pi Nt-\hat{\omega})=c\epsilon^{-a\phi}\cos(\phi-\hat{\omega}),$$

where $\phi=2\pi Nt$; that is, the period is represented by a complete revolution.

In the same way, an oscillating electromotive force will be represented by

$$E = e\epsilon^{-a\phi} \cos(\phi - \hat{\omega}).$$

Such an oscillating electromotive force for the values

$$e = 5, \quad \alpha = .1435 \text{ or } \epsilon^{-2\pi a} = .4, \quad \hat{\omega} = 0,$$

is represented in rectangular co-ordinates in Fig. 1, and in polar co-ordinates in Fig. 2. As seen from Fig. 1, the oscillating wave in rectangular co-ordinates osculates the two exponential curves

$$y = \pm e\epsilon^{-a\phi}.$$

In polar co-ordinates, the oscillating wave is represented in Fig. 2 by a spiral curve passing the zero point twice per period, and osculating the exponential spiral

$$y = \pm e\epsilon^{-a\phi}.$$

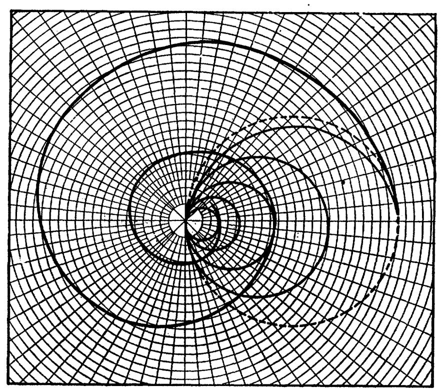

Fig. 2.

The latter is called the envelope of the oscillating wave, and is shown separately, with the same constants as Figs. 1 and 2, in

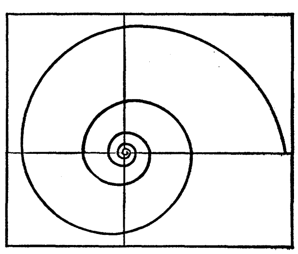

Fig. 3.

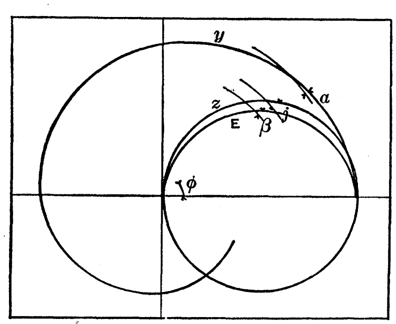

Fig. 4.

Fig. 3. Its characteristic feature is: The angle, which any concentric circle makes with the curve $y = e\epsilon^{-a\phi}$, is

$$\tan \alpha = \frac{dy}{y d\phi} = -a,$$

which is, therefore, constant, or in other words: "The envelope of the oscillating current is the loxodromic spiral, which is characterized by a constant angle of intersection with all concentric circles, or all radii vectores." The oscillating current wave is the product of the sine wave and the loxodromic spiral.

In Fig. 4 let $y = e\epsilon^{-a\phi}$ represent the loxodromic spiral;

let $z = e \cos(\phi - a)$ represent the sine wave;

and let $E = e\epsilon^{-a\phi} \cos(\phi - \omega)$ represent the oscillating wave.

We have then
$$\tan\beta = \frac{dE}{Ed\phi}$$
$$= \frac{-\sin(\phi - \hat{\omega}) - a\cos(\phi - \hat{\omega})}{\cos(\phi - \hat{\omega})}$$
$$= -\{\tan(\phi - \hat{\omega}) + a\};$$

that is, while the slope of the sine wave, $z = e\cos(\phi - \hat{\omega})$, is represented by
$$\tan\gamma = -\tan(\phi - \hat{\omega}),$$

the slope of the loxodromic spiral $y = e\epsilon^{-a\phi}$ is
$$\tan\alpha = -a = \text{constant}.$$

That of the oscillating wave $E = e\epsilon^{-a\phi}\cos(\phi - \hat{\omega})$ is
$$\tan\beta = -\{\tan(\phi - \hat{\omega}) + a\}$$

Hence, it is increased over that of the alternating sine wave by the constant a. The ratio of the amplitudes of two consequent periods is
$$A = \frac{E_{2\pi}}{E_0} = \epsilon^{-2\pi a}.$$

A is called the numerical decrement of the oscillating wave, a the exponential decrement of the oscillating wave, α the angular decrement of the oscillating wave. The oscillating wave can be represented by the equation
$$E = e\epsilon^{-\phi\tan\alpha}\cos(\phi - \hat{\omega}).$$

In the instance represented by Figs. 1 and 2, we have, $A = .4$, $a = .1435$, $\alpha = 8.2°$.

§ 2. *Impedance and Admittance.*

In complex imaginary quantities, the alternating wave

$$z = e \cos(\phi - \hat{\omega})$$

is represented by the symbol

$$F = e(\cos\hat{\omega} + j\sin\hat{\omega}) = e_1 + je_2.$$

By an extension of the meaning of this symbolic expression, the oscillating wave $E = e\epsilon^{-a\phi}\cos(\phi - \hat{\omega})$ can be expressed by the symbol

$$E = e(\cos\hat{\omega} + j\sin\hat{\omega}) \operatorname{dec}\alpha = (e_1 + je_2) \operatorname{dec}\alpha,$$

where $a = \tan\alpha$ is the exponential decrement, α the angular decrement, $\epsilon^{-2\pi a}$ the numerical decrement.

Inductance.

Let $r =$ resistance, $L =$ inductance, and $s = 2\pi NL =$ reactance.

In a circuit excited by the oscillating current,

$$C = c\epsilon^{-a\phi}\cos(\phi - \omega) = c(\cos\hat{\omega} + j\sin\hat{\omega}) \operatorname{dec}\alpha = (c_1 + jc_2) \operatorname{dec}\alpha,$$

where $c_1 = c\cos\hat{\omega}, \; c_2 = c\sin\hat{\omega}, \; a = \tan\alpha.$

We have then,

The electromotive force consumed by the resistance r of the circuit

$$E_r = rC \operatorname{dec}\alpha.$$

The electromotive force consumed by the inductance L of the circuit,

$$E_s = L\frac{dC}{dt} = 2\pi NL\frac{dC}{d\phi} = s\frac{dC}{d\phi}.$$

Hence $$E_s = -sc\epsilon^{-a\phi}\{\sin(\phi - \hat{\omega}) + a\cos(\phi - \hat{\omega})\}$$
$$= -\frac{sc\epsilon^{-a\phi}}{\cos\alpha}\sin(\phi - \hat{\omega} + \alpha).$$

Thus, in symbolic expression,

$$E_s = -\frac{sc}{\cos\alpha}\{-\sin(\hat{\omega} - \alpha) + j\cos(\hat{\omega} - \alpha)\} \operatorname{dec}\alpha$$
$$= -sc(a + j)(\cos\hat{\omega} + j\sin\hat{\omega}) \operatorname{dec}\alpha;$$

that is, $$E_s = -sC(a + j) \operatorname{dec}\alpha.$$

Hence the apparent reactance of the oscillating current circuit is, in symbolic expression,

$$S = s(a+j) \text{ dec } \alpha.$$

Hence it contains an energy component as, and the impedance is

$$U = (r - S) \text{ dec } \alpha = \{r - s(a+j)\} \text{dec } \alpha = (r - as - js) \text{ dec } \alpha.$$

Capacity.

Let r=resistance, K=capacity, and $k = \frac{1}{2\pi K}$ = capacity reactance. In a circuit excited by the oscillating current C, the electromotive force consumed by the capacity K is

$$E_k = \frac{1}{K}\int C dt = \frac{1}{2\pi NK}\int C d\phi = k \int C d\phi ;$$

or, by substitution,

$$\begin{aligned} E_k &= k\int c\epsilon^{-\alpha\phi} \cos(\phi - \hat{\omega}) d\phi \\ &= \frac{k}{1+a^2} c\epsilon^{-\alpha\phi}\{\sin(\phi - \hat{\omega}) - \alpha \cos(\phi - \hat{\omega})\} \\ &= \frac{kc\epsilon^{-\alpha\phi}}{(1+a^2)\cos\alpha} \sin(\phi - \hat{\omega} - \alpha) ; \end{aligned}$$

hence, in symbolic expression,

$$\begin{aligned} E_k &= \frac{kc}{(1+a^2)\cos\alpha}\left\{ -\sin(\hat{\omega}+\alpha) + j\cos(\hat{\omega}+\alpha)\right\} \text{dec } \alpha \\ &= \frac{kc}{1+a^2}(-a+j)(\cos\hat{\omega} + j\sin\hat{\omega}) \text{ dec } \alpha ; \end{aligned}$$

hence, $$E_k = \frac{k}{1+a^2}(-a+j) C \text{ dec } \alpha,$$

that is, the apparent capacity reactance of the oscillating circuit is, in symbolic expression,

$$K = \frac{k}{1+a^2}(-a+j) \text{ dec } \alpha.$$

We have then:

In an oscillating current circuit of resistance r, inductive reactance s, and capacity reactance k, with an exponential decrement a, the apparent impedance, in symbolic expression, is:

$$U=\left\{r-s(a+j)+\frac{k}{1+a^2}(-a+j)\right\}\operatorname{dec}\alpha,$$

$$=\left\{r-a\left(s+\frac{k}{1+a^2}\right)-j\left(s-\frac{k}{1+a^2}\right)\right\}\operatorname{dec}\alpha,$$

$$=r_a-js_a;$$

and, absolute,

$$u_a=\sqrt{r_a^2+s_a^2}$$

$$=\sqrt{\left[r-a\left(s+\frac{k}{1+a^2}\right)\right]^2+\left[s-\frac{k}{1+a^2}\right]^2}.$$

Admittance.

Let $C=c\epsilon^{-a\phi}\cos(\phi-\hat{\omega})=\text{current}.$

Then, from the preceding discussion, the electromotive force consumed by resistance r, inductive reactance s, and capacity reactance k, is

$$E=c\epsilon^{-a\phi}\left\{\cos(\phi-\hat{\omega})\left[r-as-\frac{a}{1+a^2}k\right]-\sin(\phi-\hat{\omega})\left[s-\frac{k}{1+a^2}\right]\right\}$$

$$=cu_a\epsilon^{-a\phi}\cos(\phi-\hat{\omega}+\delta),$$

where $$\tan\delta=\frac{s-\dfrac{k}{1+a^2}}{r-as-\dfrac{a}{1+a^2}k},$$

$$u_a=\sqrt{\left(s-\frac{k}{1+a^2}\right)^2+\left(r-as-\frac{a}{1+a^2}k\right)^2};$$

substituting $\hat{\omega}+\delta$ for $\hat{\omega}$, and $e=cu_a$, we have

$$E=e\epsilon^{-a\phi}\cos(\phi-\hat{\omega}),$$

$$C=\frac{e}{u_a}\epsilon^{-a\phi}\cos(\phi-\hat{\omega}-\delta)$$

$$=e\epsilon^{-a\phi}\left\{\frac{\cos\delta}{u_a}\cos(\phi-\hat{\omega})+\frac{\sin\delta}{u_a}\sin(\phi-\hat{\omega})\right\};$$

hence in complex quantities,

$$E = e(\cos\hat{\omega} + j\sin\hat{\omega})\,\text{dec}\,\alpha,$$

$$C = E\left\{\frac{\cos\delta}{u_a} + j\frac{\sin\delta}{u_a}\right\}\text{dec}\,\alpha\,;$$

or, substituting,

$$C = E\left\{\frac{r - as - \frac{a}{1+a^2}k}{\left(s - \frac{k}{1+a^2}\right)^2 + \left(r - as - \frac{a}{1+a^2}k\right)^2}\right.$$

$$\left. + j\frac{s - \frac{k}{1+a^2}}{\left(s - \frac{k}{1+a_2}\right)^2 + \left(r - as - \frac{a}{1+a^2}k\right)^2}\right\}\text{dec}\,\alpha.$$

Thus in complex quantities, for oscillating currents, we have: conductance,

$$\rho = \frac{r - as - \frac{a}{1+a^2}k}{\left(s - \frac{k}{1+a^2}\right)^2 + \left(r - as - \frac{a}{1+a^2}k\right)^2}\,;$$

susceptance,

$$\sigma = \frac{s - \frac{k}{1+a^2}}{\left(s - \frac{k}{1+a^2}\right)^2 + \left(r - as - \frac{a}{1+a^2}k\right)^2}\,;$$

admittance, in absolute values,

$$v = \sqrt{\rho^2 + \sigma^2} = \frac{1}{\sqrt{\left(s - \frac{k}{1+a^2}\right)^2 + \left(r - as - \frac{a}{1+a^2}k\right)^2}}\,;$$

in symbolic expression,

$$\Upsilon = \rho + j\sigma = \frac{\left(r - as - \frac{a}{1+a^2}k\right) + j\left(s - \frac{k}{1+a^2}\right)}{\left(s - \frac{k}{1+a^2}\right)^2 + \left(r - as - \frac{a}{1+a^2}k\right)^2}.$$

Since the impedance is

$$U = \left(r - as - \frac{a}{1+a^2} = k\right) - j\left(s - \frac{k}{1+a^2}\right) = r_a - js_a,$$

we have $\Upsilon = \frac{1}{U}$; $v = \frac{1}{u_a}$; $\rho = \frac{r_a}{u_a^2}$; $\sigma = \frac{s_a}{u_a^2}$;

that is, the same relations as in the complex quantities in alternating current circuits, except that in the present case all the constants r_a, s_a, u_a, ρ, σ, v, depend upon the decrement a.

§ 3. *Circuits of Zero Impedance.*

In an oscillating current circuit of decrement a, of resistance r, inductive resistance s, and capacity reactance k, the impedance was represented in symbolic expression by

$$U=r_a-js_a=\left(r-as-\frac{a}{1+a^2}k\right)-j\left(s-\frac{k}{1+a^2}\right),$$

or numerically by

$$u=\sqrt{r_a^2+s_a^2}=\sqrt{\left(r-as-\frac{a}{1+a^2}k\right)^2+\left(s-\frac{k}{1+a^2}\right)^2}.$$

Thus the inductive reactance s, as well as the capacity reactance k, do not represent wattless electromotive forces as in an alternating current circuit, but introduce energy components of negative sign

$$-as-\frac{a}{1+a^2}k;$$

that means,

"In an oscillating current circuit, the counter electromotive force of self-induction is not in quadrature behind the current, but lags less than 90° or a quarter period, and the charging current of a condenser is less than 90°, or a quarter period ahead of the impressed electromotive force."

In consequence of the existence of negative energy components of reactance in an oscillating current circuit, a phenomenon can exist which has no analogy in an alternating current circuit, that is, under certain conditions, the total impedance of the oscillating current circuit can equal zero:

$$U=0.$$

In this case we have

$$r-as-\frac{a}{1+a^2}k=0;\ s-\frac{k}{1+a^2}=0,$$

substituting in this equation

$$s = 2\pi NL\,;\ k = \frac{1}{2\pi NK}\,;$$

and expanding, we have

$$a = \frac{1}{\sqrt{\frac{4L}{r^2K} - 1}}\,;$$

$$2\pi N = \frac{r}{2L}\sqrt{\frac{4L}{r^2K} - 1} = \frac{r}{2aL}.$$

That is,

"If in an oscillating current circuit, the decrement $a = \frac{1}{\sqrt{\frac{4L}{r^2K} - 1}}$, and the frequency $N = \frac{r}{4\pi aL}$, the total impedance of the circuit is zero; that is, the oscillating current, when started once, will continue without external energy being impressed upon the circuit."

The physical meaning of this is: "If upon an electric circuit a certain amount of energy is impressed and then the circuit left to itself, the current in the circuit will become oscillating, and the oscillations assume the frequency $N = \frac{r}{4\pi aL}$, and the decrement $a = \frac{1}{\sqrt{\frac{4L}{r^2K} - 1}}$.

That is, the oscillating currents are the phenomena by which an electric circuit of disturbed equilibrium returns to equilibrium.

This feature shows the origin of the oscillating currents, and the means to produce such currents by disturbing the equilibrium of the electric circuit, for instance by the discharge of a condenser, by make and break of the circuit, by sudden electrostatic charge, as lightning, etc. Obviously, the most important oscillating currents are those flowing in a circuit of zero impedance, representing oscillating discharges of the circuit. Lightning strokes usually belong to this class.

§ 4. *Oscillating Discharges.*

The condition of an oscillating discharge is $U = 0$; that is:

$$a = \frac{1}{\sqrt{\frac{4L}{r^2K} - 1}}, \qquad 2\pi N = \frac{r}{2aL} = \frac{r}{2L}\sqrt{\frac{4L}{r^2K} - 1}.$$

If $r=0$, that is, in a circuit without resistance, we have $a=0$, $N=\frac{1}{2\pi\sqrt{LK}}$; that is, the currents are alternating with no decrement, and the frequency is that of resonance.

If $\frac{4L}{r^2K}-1<0$, that is, $r>2\sqrt{\frac{L}{K}}$, a and N become imaginary; that is, the discharge ceases to be oscillatory. An electrical discharge assumes an oscillating nature only, if $r<2\sqrt{\frac{L}{K}}$. In the case $r=2\sqrt{\frac{L}{K}}$ we have $a=\infty$, $N=0$; that is, the current dies out without oscillation.

From the foregoing we have seen that oscillating discharges, — as for instance the phenomena taking place if a condenser charged to a given potential is discharged through a given circuit, or if lightning strikes the line circuit, — is defined by the equation: $U=0 \text{ dec } a$.

Since $C=(c_1+jc_2) \text{ dec } \alpha, \quad E_r=Cr \text{ dec } \alpha,$

$$E_s=-sC(a+j) \text{ dec } \alpha, \quad E_k=\frac{k}{1+a^2}C(-a+j) \text{ dec } \alpha,$$

we have

$$r-as-\frac{a}{1+a^2}k=0,$$

$$-s+\frac{k}{1+a^2}=0;$$

hence, by substitution,

$$E_k=sC(-a+j) \text{ dec } \alpha.$$

The two constants, c_1 and c_2, of the discharge, are determined by the initial conditions, that is, the electromotive force and the current at the time $t=0$.

Let a condenser of capacity K be discharged through a circuit of resistance r and inductance L. Let e = electromotive force at the condenser in the moment of closing the circuit, that is, at the time $t=0$ or $\phi=0$. At this moment the current is zero, that is,

$$C=jc_2, \quad c_1=0.$$

Since $E_k=sC(-a+j) \text{ dec } \alpha=e$ at $\phi=0$,

we have $sc_2\sqrt{1+a^2}=e$ or $c_2=\frac{e}{s\sqrt{1+a^2}}.$

Substituting this, we have,

$$C = j\frac{e}{s\sqrt{1+a^2}}\operatorname{dec}\alpha, \qquad E_r = je\frac{r}{s\sqrt{1+a^2}}\operatorname{dec}\alpha,$$

$$E_s = \frac{e}{\sqrt{1+a^2}}(1-ja)\operatorname{dec}\alpha, \quad E_k = -\frac{e}{\sqrt{1+a^2}}(1+ja)\operatorname{dec}\alpha,$$

the equations of the oscillating discharge of a condenser of initial voltage e.

Since

$$s = 2\pi NL,$$

$$a = \frac{1}{\sqrt{\frac{4L}{r^2K}-1}},$$

$$2\pi N = \frac{r}{2aL},$$

we have

$$s = \frac{r}{2a} = \frac{r}{2}\sqrt{\frac{4L}{r^2K}-1};$$

hence, by substitution,

$$C = je\sqrt{\frac{K}{L}}\operatorname{dec}\alpha, \quad E^r = jer\sqrt{\frac{K}{L}}\operatorname{dec}\alpha,$$

$$E_s = \frac{er}{2}\sqrt{\frac{K}{L}}\left(\sqrt{\frac{4L}{r^2K}-1}-j\right)\operatorname{dec}\alpha,$$

$$E_k = -\frac{er}{2}\sqrt{\frac{K}{L}}\left(\sqrt{\frac{4L}{r^2K}-1}+j\right)\operatorname{dec}\alpha,$$

$$a = \frac{1}{\sqrt{\frac{4L}{r^2K}-1}}, \quad N = \frac{r\sqrt{\frac{4L}{r^2K}-1}}{4\pi L},$$

the final equations of the oscillating discharge, in symbolic expression.

§ 5. *Numerical Examples.*

To get an estimate of the numerical values of the constants of oscillating discharges, some cases may be investigated.

A. VERY HIGH FREQUENCY.

A short, straight conductor may be terminated by two balls. The balls represent the capacity, while the conductor represents the resistance and inductance. Without entering into the exact calculation, let the resistance of the conductor

$r = .0001$ ohms; the inductance of the conductor $L = .00025$ millihenry; the capacity of the two balls $K = 10^{-6}$ microfarads. We have then

$$a = 10^{-7}; \text{ angle } \alpha = 5.7 \times 10^{-6}; \; N = 32 \times 10^4;$$

that is, 320,000,000 cycles per second.

The amplitude is reduced to $\frac{1}{100}$ after the time t_0, or angle ϕ_0, where

$$\epsilon^{-a\phi_0} = .01; \; \phi_0 = 2\pi N t_0;$$

hence,
$$\phi_0 = 4.6 \times 10^7; \; t_0 = .023;$$

that is, after .023 second the oscillation has practically died out; that is, decreased to $\frac{1}{100}$ of its amplitude, after making 7,400,000 complete oscillations.

The equations of the oscillating discharge are in this case, at $e = 10{,}000$ volts initial charge,

$$C = 20j \operatorname{dec} \alpha; \; E_r = .002j \operatorname{dec} \alpha.$$
$$E_s = (10{,}000 - .001j) \operatorname{dec} \alpha; \; E_k = -(10{,}000 + .001j) \operatorname{dec} \alpha.$$

As seen, for a moment 20 amperes flow at a potential of 10,000 volts, representing an instantaneous flow of about 100 K.W.

To make the discharge non-oscillatory, the resistance would have to be $r > 2\sqrt{\frac{L}{K}} > 1000$ ohms; that is, more than 10,000,000 times as much as assumed. That is, a wet string, or similar conductor, will not allow electrical oscillation.

B. Underground Circuit of four Miles in Length, of two Lead-covered Cables consisting of Wire No. 00, B. & S.

$$r = 3.3 \text{ ohms}; \; L = 7.5 \text{ microhenrys}; \; K = 1.2 \text{ microfarads};$$

hence,
$$a = .021; \; N = 1670; \; t_0\phi = .021;$$

that is, in .021 seconds, or after 35 cycles, the oscillation has died out to $\frac{1}{100}$ of its initial value. At $e = 2000$ volts initial charge, the equations of phenomenon will be

$$C = 25.3j \operatorname{dec} \alpha; \; E_r = 83.5j \operatorname{dec} \alpha;$$
$$E_s = (2000 - 42j) \operatorname{dec} \alpha \;\; E_k = -(2000 + 42j) \operatorname{dec} \alpha.$$

The phenomenon will become non-oscillatory; that is,

$$N = 0; \; a = \infty; \text{ for } r = 158 \text{ ohms}.$$

that is, in such cables electric oscillations can take place at comparatively moderate frequency, and of considerable duration.

C. Transatlantic Cable.

Assuming approximately, $r = 40{,}000$ ohms, $L = 30$ h., $K = 1300$ microfarads; we then have $r < 300$, the condition under which electrical oscillations can take place.

If the resistance were $\frac{1}{200}$ the value it is in reality, that is, if we had $r = 200$ ohms, we could have $a = 0.88$, $N = 0.6$, and in this case, at $e = 100$ volts initial charge, the equations of the phenomenon would be, $C = 0.658j \operatorname{dec} \alpha$, $E_r = 131.6j \operatorname{dec} \alpha$, $E_s = (75 - 66j) \operatorname{dec} \alpha$, $E_k = -(75 + 66j) \operatorname{dec} \alpha$; that is, in a transatlantic cable electrical oscillations cannot take place, due to its high resistance. If, however, the resistance were low enough to permit electrical oscillations, that is, less than $\frac{1}{133}$ of what it is in reality, the oscillations would take place extremely slowly, each complete oscillation occupying more than one second.

In reality, due to the capacity not being centralized in a condenser, but distributed over the whole circuit, the phenomenon is more complex, and has to be investigated on the lines of circuits containing distributed capacity.

We see, however, from these instances the enormous range of frequencies, at which electrical oscillations take place, from frequencies of hundred millions of cycles per second to a frequency of more than a second per cycle.

§ 6. *Oscillating Current Transformer.*

As an instance of the application of the symbolic method of analyzing the phenomena caused by oscillating currents, the transformation of such currents may be investigated. If an oscillating current is produced in a circuit including the primary of a transformer, oscillating currents will also flow in the secondary of this transformer. In a transformer let the ratio of secondary to primary turns be p. Let the secondary be closed by a circuit of total resistance, $r_1 = r_1' + r_1''$, where $r_1' =$ external, $r_1'' =$ internal, resistance. The total inductance $L_1 = L_1' + L_1''$, where $L_1' =$ external, $L_1'' =$ internal inductance, total capacity, K_1. Then the total admittance of the secondary circuit is

$$Y_1 = (\rho_1 + j\sigma_1) \operatorname{dec} \alpha = \frac{1}{\left(r_1 - as_1 - \frac{a}{1+a^2} k_1\right) - j\left(s_1 - \frac{k}{1+a^2}\right)},$$

where $s_1 = 2\pi N L_1 =$ inductive reactance; $k_1 = \frac{1}{2\pi N K} =$ capacity reactance. Let $r_0 =$ effective hysteretic resistance, $L_0 =$ inductance; hence, $s_0 = 2\pi N L_0 =$ reactance; hence,

$$Y_0 = \rho_0 + j\sigma_0 = \frac{1}{(r_0 - as_0) - js_0} = \text{admittance}$$

of the primary exciting circuit of the transformer; that is, the admittance of the primary circuit at open secondary circuit.

As discussed elsewhere, a transformer can be considered as consisting of the secondary circuit supplied by the impressed electromotive force over leads, whose impedance is equal to the sum of primary and secondary transformer impedance, and which are shunted by the exciting circuit, outside of the secondary, but inside of the primary impedance.

Let $r=$ resistance; $L=$ inductance; $K=$ capacity; hence,

$$s = 2\pi NL = \text{inductive reactance},$$

$k = \frac{1}{2\pi NK} =$ capacity reactance of the total primary circuit, including the primary coil of the transformer. If $E_1' = E_1' \operatorname{dec} \alpha$ denotes the electromotive force induced in the secondary of the transformer by the mutual magnetic flux; that is, by the oscillating magnetism interlinked with the primary and secondary coil we have $C_1 = E_1 \Upsilon_1 \operatorname{dec} \alpha =$ secondary current.

Hence, $C_1' = p,\ C_1 \operatorname{dec} \alpha = p,\ E' \Upsilon_1 \operatorname{dec} \alpha =$ primary load current, or component of primary current corresponding to secondary current. Also, $C_0 = \frac{p'}{p} E_1' \Upsilon_0 \operatorname{dec} \alpha =$ primary exciting current; hence, the total primary current is

$$C = C_1' + C_0 = \frac{E_1'}{p} \{\Upsilon_0 + p^2 \Upsilon_1\} \operatorname{dec} \alpha.$$

$E' = \frac{E_1'}{p} \operatorname{dec} \alpha =$ induced primary electromotive force. Hence the total primary electromotive force is

$$E = (E' + CU) \operatorname{dec} \alpha = \frac{E_1'}{p} \{1 + U\Upsilon_0 + p^2 U \Upsilon_1\} \operatorname{dec} \alpha.$$

In an oscillating discharge the total primary electromotive force $E=0$; that is,

$$1 + U\Upsilon_0 + p^2 U \Upsilon_1 = 0;$$

or, the substitution

$$1 + \frac{\left(r - as - \frac{a}{1+a^2}k\right) - j\left(s - \frac{k}{1+a^2}\right)}{(r_0 - as_0) - js_0} + p^2 \frac{\left(r - as - \frac{a}{1+a^2}k\right) - j\left(s - \frac{k}{1+a^2}\right)}{\left(r_1 - as_1 - \frac{a}{1+a^2}k_1\right) - j\left(s_1 - \frac{k_1}{1+a^2}\right)} = 0.$$

Substituting in this equation, $s = 2\pi NK$, $k = \frac{1}{2\pi NK}$, etc., we get a complex imaginary equation with the two constants a and N. Separating this equation in the real and the imaginary parts, we derive two equations, from which the two constants a and N of the discharge are calculated.

If the exciting current of the transformer is negligible; that is, if $\Upsilon_0 = 0$, the equation becomes essentially simplified:

$$1 + p^2 \frac{\left(r - as - \frac{a}{1+a^2}k\right) - j\left(s - \frac{k}{1+a^2}\right)}{\left(r_1 - as_1 - \frac{a}{1+a^2}k_1\right) - j\left(s_1 - \frac{k_1}{1+a^2}\right)} = 0;$$

that is,

$$\left(r_1 - as_1 - \frac{a}{1+a^2}k_1\right) + p^2\left(r - as - \frac{a}{1+a^2}k\right) = 0;$$

$$\left(s_1 - \frac{k_1}{1+a^2}\right) + p^2\left(s - \frac{k}{1+a^2}\right) = 0;$$

or, combined:

$$(r_1 - 2as_1) + p^2(r - 2as) = 0,$$

$$r_1 + p^2 r = 2a(s_1 + p^2 s),$$

$$k_1 + p^2 k = (1 + a^2)(s_1 + p^2 s).$$

Substituting for s_1, s, k_1, k, we have

$$a = \frac{1}{\sqrt{\frac{4(L_1 + p^2 L)}{(r_1 + p^2 r)^2 (K_1 + p^2 K)} - 1}},$$

$$2\pi N = \frac{r_1 + p^2 r}{2a(L_1 + p^2 L)} = \frac{r_1 + p^2 r}{2(L_1 + p^2 L)} \sqrt{\frac{4(L_1 + p^2 L)}{(r_1 + p^2 r)^2 (K_1 + p^2 K)} - 1},$$

$$E = \frac{E_1'}{p}\{1 + p^2 U \Upsilon_1\} \operatorname{dec} \alpha,$$

$$C = pE_1' \Upsilon_1 \operatorname{dec} \alpha,$$

$$C_1 = E_1' \Upsilon_1 \operatorname{dec} \alpha,$$

the equations of the oscillating current transformer, with E_1' as parameter.

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